

# A Multilevel Newton Iteration Method for Eigenvalue Problems\*

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## Abstract

We propose a new type of multilevel method for solving eigenvalue problems based on Newton iteration. With the proposed iteration method, solving eigenvalue problem on the finest finite element space is replaced by solving a small scale eigenvalue problem in a coarse space and solving a series of augmented linear problems, derived by Newton step in the corresponding series of finite element spaces. This iteration scheme improves overall efficiency of the finite element method for solving eigenvalue problems. Finally, some numerical examples are provided to validate the efficiency of the proposed numerical scheme.

**Keywords.** Eigenvalue problem, finite element method, Newton's method, multilevel iteration.

## 1 Introduction

The original purpose of Newton's method is to seek the root of an equation. With a suitable initial guess, Newton iteration is usually convergent. Furthermore, the convergence is at least quadratic in a neighborhood of a simple root. So Newton's iteration is an extremely powerful technique in numerical methods. Nowadays

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Newton's method is widely applied to minimization and maximization problems, multiplicative inverses of numbers and power series, solving transcendent equations, complex functions, nonlinear systems of equations.

Our work is to design a Newton's method to solve PDE eigenvalue problems. Taking advantage of the rapid convergence of Newton's method, we design a Newton's method to solve eigenvalue problems, treating eigenvalue problem as a nonlinear equation. Some works [6, 13, 15, 16] had exploited the Newton's method for eigenvalue problems. The Newton's method (see [6]) is based on an approximate eigenpair  $(\lambda_0, x_0)$  and wishes to determine  $\delta\lambda$  and  $\delta x$  so that  $(\lambda_0 + \delta\lambda, x_0 + \delta x)$  is an improved approximation of the exact eigenpair.

In recent decades, the study of solving large scale eigenvalue problems, arising from modern science and engineering society, has become one of the major focuses of numerical analysts and engineers. However, it is always a difficult task to solve high-dimensional eigenvalue problems which come from physical and chemical sciences. About the solution of eigenvalue problems, [3, 8, 7, 10, 11, 14] and the references cited therein give some types of multilevel or multigrid schemes.

The aim of this paper is to present a type of multilevel iteration scheme based on Newton's method for eigenvalue problems. The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [1, 2], Chatelin [5] and references cited therein. Here we adopt some basic results in these papers for our analysis. The corresponding error and complexity discussion of the proposed iteration scheme for the eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an optimal computational work when we can solve the associated augmented linear problems with the optimal complexity. Although the Newton's method is sensitive to initial guess, we use multilevel technique to overcome this difficulty. Since it is easy to find a good approximation in the coarse grid, which provides a good initial guess for the fine grid, the Newton type iteration method is reasonable. According to the theory for mixed finite element method, we prove the existence and the uniqueness of the solution to the proposed scheme.

This paper is organized as follows. In Section 2, we introduce the finite element method for the eigenvalue problem and give the corresponding basic error estimates. A type of one Newton iteration step is presented and the error estimates of the proposed scheme are analyzed in Section 3. In Section 4, we propose a type of multilevel iteration scheme for multi eigenvalues solving. The computational work estimate of the multilevel iteration method is discussed in Section 5. In Section 6, two numerical examples are presented to validate our theoretical analysis. Some concluding remarks are provided in the final section.

## 2 Finite element method for eigenvalue problems

In this section, we introduce some notation and error estimates of the finite element approximation for the eigenvalue problem. The letter  $C$  (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols  $\lesssim$ ,  $\gtrsim$  and  $\approx$  will be used in this paper. These  $x_1 \lesssim y_1, x_2 \gtrsim y_2$  and  $x_3 \approx y_3$ , mean that  $x_1 \leq C_1 y_1$ ,  $x_2 \geq c_2 y_2$  and  $c_3 x_3 \leq y_3 \leq C_3 x_3$  for some constants  $C_1, c_2, c_3$  and  $C_3$  that are independent of mesh sizes (see, e.g., [17]).

In our methodology description, we are concerned with the following model problem:

Find  $(\lambda, u) \in \mathbb{R} \times V$  such that  $b(u, u) = 1$  and

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V, \quad (2.1)$$

where  $V := H_0^1(\Omega)$ ,  $a(\cdot, \cdot)$  and  $b(\cdot, \cdot)$  are bilinear forms defined by

$$a(u, v) = \int_{\Omega} \nabla u \nabla v d\Omega, \quad b(u, v) = \int_{\Omega} u v d\Omega.$$

In this paper, based on these two bilinear forms, we define the norms  $\|\cdot\|_a$  and  $\|\cdot\|_b$  as follows

$$\|v\|_a^2 = a(v, v), \quad \|v\|_b^2 = b(v, v).$$

It is well known that the norm  $\|\cdot\|_a$  is a norm in the space  $V$  and  $\|\cdot\|_b$  is a norm in the space  $L^2(\Omega)$ .

For the eigenvalue  $\lambda$ , there exists the following Rayleigh quotient expression (see, e.g., [1, 2, 18])

$$\lambda = \frac{a(u, u)}{b(u, u)}.$$

From [2, 5], we know the eigenvalue problem (2.1) has an eigenvalue sequence  $\{\lambda_j\}$ :

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \rightarrow \infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where  $b(u_i, u_j) = \delta_{ij}$ ,  $\delta_{ij}$  is Kronecker notation. In the sequence  $\{\lambda_j\}$ , the  $\lambda_j$  are repeated according to their geometric multiplicity. In order to give the error estimates, let  $M(\lambda_i)$  denote the eigenfunction space corresponding to the eigenvalue  $\lambda_i$  which is defined by

$$M(\lambda_i) = \{w \in V : w \text{ is an eigenfunction of (2.1) corresponding to } \lambda_i\}.$$

Now, let us define the finite element approximations of the problem (2.1). First we generate a shape-regular decomposition of the computing domain  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) into triangles or rectangles for  $d = 2$  (tetrahedrons or hexahedrons for  $d = 3$ ). The diameter of a cell  $K \in \mathcal{T}_h$  is denoted by  $h_K$ . The mesh diameter  $h$  describes the maximum diameter of all cells  $K \in \mathcal{T}_h$ . Based on the mesh  $\mathcal{T}_h$ , we can construct the linear finite element space denoted by  $V_h \subset V$ . We assume that the finite element space  $V_h$  satisfies the following assumption:

For any  $w \in V$

$$\lim_{h \rightarrow 0} \inf_{v_h \in V_h} \|w - v_h\|_a = 0. \quad (2.2)$$

The finite element approximation for (2.1) is defined as follows: Find  $(\bar{\lambda}_h, \bar{u}_h) \in \mathbb{R} \times V_h$  such that  $b(\bar{u}_h, \bar{u}_h) = 1$  and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \quad (2.3)$$

From (2.3), we know the following Rayleigh quotient expression for  $\bar{\lambda}_h$  holds (see, e.g., [1, 2, 18])

$$\bar{\lambda}_h = \frac{a(\bar{u}_h, \bar{u}_h)}{b(\bar{u}_h, \bar{u}_h)}.$$

Similarly, we know from [2, 5] the eigenvalue problem (2.3) has eigenvalues

$$0 < \bar{\lambda}_{1,h} \leq \bar{\lambda}_{2,h} \leq \cdots \leq \bar{\lambda}_{k,h} \leq \cdots \leq \bar{\lambda}_{N_h,h},$$

and the corresponding eigenfunctions

$$\bar{u}_{1,h}, \bar{u}_{2,h}, \cdots, \bar{u}_{k,h}, \cdots, \bar{u}_{N_h,h},$$

where  $b(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}$ ,  $1 \leq i, j \leq N_h$  ( $N_h$  is the dimension of the finite element space  $V_h$ ).

From the minimum-maximum principle (see, e.g., [1, 2]), the following upper bound result holds

$$\lambda_i \leq \bar{\lambda}_{i,h}, \quad i = 1, 2, \cdots, N_h.$$

Similarly, let  $M_h(\lambda_i)$  denote the approximate eigenfunction space corresponding to the eigenvalue  $\lambda_i$  which is defined by

$$M_h(\lambda_i) = \left\{ w_h \in V_h : w_h \text{ is an eigenfunction of (2.3) corresponding to } \lambda_i \right\}.$$

From [1, 2], each eigenvalue  $\bar{\lambda}_{i,h}$  can be defined as follows

$$\bar{\lambda}_{i,h} = \inf_{\substack{v_h \in V_h \\ v_h \perp M_h(\lambda_j) \text{ for } \lambda_j < \lambda_i}} \frac{a(v_h, v_h)}{b(v_h, v_h)}. \quad (2.4)$$

In order to give the error estimate result for the eigenvalue problems by the finite element method, we define

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i), \|w\|_a=1} \inf_{v_h \in V_h} \|w - v_h\|_a, \quad (2.5)$$

and  $\eta_a(h)$  as

$$\eta_a(h) = \sup_{f \in V, \|f\|_b=1} \inf_{v_h \in V_h} \|Tf - v_h\|_a, \quad (2.6)$$

where the operator  $T : V' \rightarrow V$  is defined as

$$a(Tf, v) = b(f, v), \quad \forall f \in V', \quad \forall v \in V.$$

There exist the following error estimates for the eigenpair approximations by finite element method.

**Proposition 2.1.** (*[1, Lemma 3.7, (3.29b)], [2, P. 699] and [5]*)

(i) *For any eigenfunction approximation  $\bar{u}_{i,h}$  of (2.3) ( $i = 1, 2, \dots, N_h$ ), there is an eigenfunction  $u_i$  of (2.1) corresponding to  $\lambda_i$  such that  $\|u_i\|_b = 1$  and*

$$\|u_i - \bar{u}_{i,h}\|_a \leq C\delta_h(\lambda_i). \quad (2.7)$$

Furthermore,

$$\|u_i - \bar{u}_{i,h}\|_b \leq C\eta_a(h)\|u_i - \bar{u}_{i,h}\|_a. \quad (2.8)$$

(ii) *For each eigenvalue, we have*

$$\lambda_i \leq \bar{\lambda}_{i,h} \leq \lambda_i + C\delta_h^2(\lambda_i). \quad (2.9)$$

Here and hereafter  $C$  is some constant depending on  $\lambda_i$  but independent of the mesh size  $h$ .

### 3 A Newton iteration method for eigenvalue problem

The aim of this section is to present a type of one Newton iteration step to improve the accuracy of the given eigenpair approximations. This iteration method only contains solving augmented linear problems in a finer finite element space. Here we only state the numerical method for the first and simple eigenvalue. In the next section, we will show the case of multi eigenvalues.

For the analysis in this paper, we introduce the error expansion of the eigenvalue by the Rayleigh quotient formula which comes from [1, 2, 12, 18].

**Lemma 3.1** ([1, 2, 12, 18]). Assume  $(\bar{\lambda}_h, \bar{u}_h)$  is a true solution of the eigenvalue problem (2.3) and  $0 \neq \psi_h \in V_h$ . Let us define

$$\hat{\lambda}_h = \frac{a(\psi_h, \psi_h)}{b(\psi_h, \psi_h)}.$$

Then we have

$$\hat{\lambda}_h - \bar{\lambda}_h = \frac{a(\bar{u}_h - \psi_h, \bar{u}_h - \psi_h)}{b(\psi_h, \psi_h)} - \bar{\lambda}_h \frac{b(\bar{u}_h - \psi_h, \bar{u}_h - \psi_h)}{b(\psi_h, \psi_h)}.$$

### 3.1 Newton iteration for eigenvalue problem

This subsection introduces the main idea that deduces our numerical method in this paper. Here, we use the Newton iteration idea to solve the eigenproblem (2.1):

Find  $(\lambda, u) \in \mathbb{R} \times V$  such that

$$\begin{cases} a(u, v) - \lambda b(u, v) = 0, & \forall v \in V, \\ b(u, u) - 1 = 0. \end{cases} \quad (3.1)$$

If we have an eigenpair approximation  $(\mu_0, u_0)$  with  $b(u_0, u_0) = 1$ , the Newton iteration method for (3.1) is to find  $(\tilde{\lambda}, \tilde{u}) \in \mathbb{R} \times V$  such that

$$\begin{cases} a(\tilde{u} - u_0, v) - \mu_0 \cdot b(\tilde{u} - u_0, v) - (\tilde{\lambda} - \mu_0)b(u_0, v) = \\ \quad -(a(u_0, v) - \mu_0 \cdot b(u_0, v)), & \forall v \in V, \\ -b(\tilde{u} - u_0, u_0) = 0. \end{cases} \quad (3.2)$$

After simplifying (3.2), we have the following equation for the new eigenpair approximation  $(\tilde{\lambda}, \tilde{u}) \in \mathbb{R} \times V$

$$\begin{cases} a(\tilde{u}, v) - \mu_0 \cdot b(\tilde{u}, v) - \tilde{\lambda}b(u_0, v) = -\mu_0b(u_0, v), & \forall v \in V, \\ -b(\tilde{u} - u_0, u_0) = 0. \end{cases} \quad (3.3)$$

Now, we come to prove that the mixed problem (3.3) has only one solution. For this aim, we define the following bilinear forms

$$A_{\mu_0}(u, v) = a(u, v) - \mu_0b(u, v), \quad B(v, \nu) = -\nu b(u_0, v), \quad (3.4)$$

where  $u \in V, v \in V, \nu \in W = \mathbb{R}$  and  $\mu_0 = \frac{a(u_0, u_0)}{b(u_0, u_0)}$ .

Assume that  $f \in V'$  and  $g \in W'$ . We consider the following mixed problem:

Find  $(u, \lambda) \in V \times W$  such that

$$\begin{cases} A_{\mu_0}(u, v) + B(v, \lambda) = f(v), & \forall v \in V, \\ B(u, \nu) = g(\nu), & \forall \nu \in W. \end{cases} \quad (3.5)$$

About the existence and uniqueness of problem (3.5), the following theorem holds.

**Theorem 3.1.** Assume  $u_0$  is an eigenfunction approximation to  $M(\lambda_1)$  with sufficiently small error and  $\|u_0\|_b = 1$ . Then the bilinear forms defined in (3.4) satisfy the following conditions

1. There exists  $\alpha > 0$  (depends on  $\lambda_2 - \lambda_1$ ) such that

$$A_{\mu_0}(v, v) \geq \alpha \|v\|_a^2, \quad \forall v \in V_0, \quad (3.6)$$

where  $V_0 = \{v | B(v, \nu) = 0, \forall \nu \in W\} = \{v | b(u_0, v) = 0\}$ .

2. There exists  $\sigma > 0$  (depends on  $1/\mu_0$ ) such that

$$\sup_{v \in V} \frac{B(v, \nu)}{\|v\|_a} \geq \sigma |\nu|, \quad \forall \nu \in W. \quad (3.7)$$

Based on these two conditions, the mixed equation (3.5) has only one solution.

*Proof.* We decompose  $u_0$  as  $u_0 = w_1 + w_1^\perp$  such that  $w_1 \in M(\lambda_1)$  and  $w_1^\perp \perp M(\lambda_1)$ .

Since  $u_0$  ( $\|u_0\|_b = 1$ ) is an eigenfunction approximation to  $M(\lambda_1)$  with sufficiently small error, there exists a small enough number  $\delta$  such that

$$\|u_0 - w_1\|_a \leq \delta. \quad (3.8)$$

From Lemma 3.1, we also have

$$|\mu_0 - \lambda_1| \leq C\delta^2. \quad (3.9)$$

Since (3.8) and  $\|u_0\|_b^2 = \|w_1\|_b^2 + \|w_1^\perp\|_b^2$ ,  $w_1^\perp$  and  $w_1$  have estimates

$$\|w_1^\perp\|_b \leq C\|w_1^\perp\|_a \leq C\delta, \quad \|w_1\|_b \geq 1 - C\delta.$$

We also do the decomposition  $v = v_1 + v_1^\perp$  with  $v_1 \in M(\lambda_1)$  and  $v_1^\perp \perp M(\lambda_1)$  for  $v \in V_0$ . Since  $b(w_1 + w_1^\perp, v_1 + v_1^\perp) = 0$ , we have

$$\|v_1\|_b \|w_1\|_b = |b(v_1, w_1)| = |-b(v_1^\perp, w_1^\perp)| = |-b(v, w_1^\perp)| \leq C\delta \|v\|_b.$$

Then  $\|v_1\|_b$  has the following estimate

$$\|v_1\|_b \leq \frac{C\delta}{1 - C\delta} \|v\|_b \leq C\delta \|v\|_b. \quad (3.10)$$

From (3.10) and the property  $\|v\|_b^2 = \|v_1\|_b^2 + \|v_1^\perp\|_b^2$ , the following estimates hold

$$\begin{aligned} b(v, v) &= b(v_1, v_1) + b(v_1^\perp, v_1^\perp) \\ &\leq C\delta^2 b(v, v) + \frac{1}{\lambda_2} a(v_1^\perp, v_1^\perp) \end{aligned}$$

$$\leq C\delta^2 b(v, v) + \frac{1}{\lambda_2} a(v, v).$$

Thus we have the following inequality

$$b(v, v) \leq \frac{1}{\lambda_2(1 - C\delta^2)} a(v, v). \quad (3.11)$$

From (3.9), (3.11) and the definition of  $A_\mu(\cdot, \cdot)$ , the following inequalities hold

$$\begin{aligned} a(v, v) - \mu_0 b(v, v) &\geq \left(1 - \frac{\mu_0}{\lambda_2(1 - C\delta^2)}\right) a(v, v) \\ &\geq \frac{\lambda_2(1 - C\delta^2) - \mu_0}{\lambda_2(1 - C\delta^2)} a(v, v) \\ &\geq \frac{\lambda_2 - \lambda_1 - C\delta^2}{\lambda_2(1 - C\delta^2)} a(v, v). \end{aligned}$$

It means (3.6) holds for  $\alpha = (\lambda_2 - \lambda_1 - C\delta^2)/(\lambda_2(1 - C\delta^2))$  when  $\delta$  is small enough.

Now, we come to prove (3.7). From the definitions of  $B(\cdot, \cdot)$  and  $\mu$ , we have

$$\sup_{v \in V} \frac{B(v, \nu)}{\|v\|_a} \geq |\nu| \frac{b(u_0, u_0)}{\|u_0\|_a} = \frac{|\nu|}{\mu_0}, \quad \forall \nu \in W.$$

It means that (3.7) holds for

$$\sigma = \frac{1}{\mu_0}.$$

From the theory for the mixed finite element method [4], there exists only one solution for the equation (3.5).  $\square$

**Corollary 3.1.** *Under the conditions of Theorem 3.1, the following inequality holds*

$$\|w\|_a + |\gamma| \leq C_4 \sup_{0 \neq (v, \nu) \in V \times W} \frac{A_{\mu_0}(w, v) + B(v, \gamma) + B(w, \nu)}{\|v\|_a + |\nu|}, \quad (3.12)$$

for any  $(w, \gamma) \in V \times W$ . The constant  $C_4$  depends on  $1/(\lambda_2 - \lambda_1)$  and  $\lambda_1$ .

## 3.2 One Newton iteration step

Based on the discussion in the last subsection, we propose an one correction step to improve the given eigenpair approximation. Assume we have obtained an eigenpair approximation  $(\lambda_{1,h_k}, u_{1,h_k}) \in \mathbb{R} \times V_{h_k}$  with  $\|u_{1,h_k}\|_b = 1$ . Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximation  $(\lambda_{1,h_k}, u_{1,h_k})$ . Let  $V_{h_{k+1}} \subset V$  be a finer finite element space such that  $V_{h_k} \subset V_{h_{k+1}}$ . Based on this finer finite element space, we define the following one Newton iteration step.



**Algorithm 3.1.** *One Newton Iteration Step*

1. *Solve the augmented mixed problem:*

Find  $(\widehat{\lambda}_{1,h_{k+1}}, \widehat{u}_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$  such that

$$\begin{cases} a(\widehat{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(\widehat{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \widehat{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}) \\ \quad = -\lambda_{1,h_k} b(u_{1,h_k}, v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\widehat{u}_{1,h_{k+1}}, u_{1,h_k}) = b(u_{1,h_k}, u_{1,h_k}). \end{cases} \quad (3.13)$$

2. *Do the normalization for  $\widehat{u}_{1,h_{k+1}}$  as*

$$u_{1,h_{k+1}} = \frac{\widehat{u}_{1,h_{k+1}}}{\|\widehat{u}_{1,h_{k+1}}\|_b} \quad (3.14)$$

*and compute the Rayleigh quotient for  $u_{1,h_{k+1}}$*

$$\lambda_{1,h_{k+1}} = \frac{a(u_{1,h_{k+1}}, u_{1,h_{k+1}})}{b(u_{1,h_{k+1}}, u_{1,h_{k+1}})}. \quad (3.15)$$

Then we obtain a new eigenpair approximation  $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$ . Summarize the above two steps into

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \text{Newton\_Iteration}(\lambda_{1,h_k}, u_{1,h_k}, V_{h_{k+1}}).$$

**Theorem 3.2.** Assume  $(\lambda_{1,h_k}, u_{1,h_k})$  is a good enough approximation to  $(\lambda_1, u_1)$  such that (3.6), (3.7) hold and  $\lambda_{1,h_k} = a(u_{1,h_k}, u_{1,h_k})/b(u_{1,h_k}, u_{1,h_k})$ . After one iteration step, the resultant approximation  $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$  has the following error estimates

$$\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \leq C_5 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2, \quad (3.16)$$

$$|\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_{k+1}}| \leq C_6 \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^4, \quad (3.17)$$

where  $C_5$  and  $C_6$  are constants which depend on  $1/(\lambda_2 - \lambda_1)$  and  $\lambda_1$  but are independent of the mesh sizes  $h_k$  and  $h_{k+1}$ .

*Proof.* From the definition (2.3), we know the eigenpair approximation  $(\bar{\lambda}_{1,h_{k+1}}, \bar{u}_{1,h_{k+1}})$  satisfies the following equations

$$\begin{cases} a(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \bar{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}) \\ \quad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \bar{\lambda}_{1,h_{k+1}} b(u_{1,h_k}, v_{h_{k+1}}), \\ \quad \forall v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\bar{u}_{1,h_{k+1}}, u_{1,h_k}) = b(\bar{u}_{1,h_{k+1}}, u_{1,h_k}). \end{cases} \quad (3.18)$$

Let us define  $w_{h_{k+1}} := \bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}$  and  $\gamma := \bar{\lambda}_{1,h_{k+1}} - \hat{\lambda}_{1,h_{k+1}}$ . From (3.13) and (3.18), the following equations hold

$$\begin{cases} a(w_{h_{k+1}}, v_{h_{k+1}}) - \lambda_{1,h_k} b(w_{h_{k+1}}, v_{h_{k+1}}) - \gamma b(u_{1,h_k}, v_{h_{k+1}}) \\ \quad = (\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}) b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, v_{h_{k+1}}), & \forall v_{h_{k+1}} \in V_{h_{k+1}}, \\ \nu b(w_{h_{k+1}}, u_{1,h_k}) = \nu b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, u_{1,h_k}) \\ \quad = -\nu \frac{1}{2} b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, \bar{u}_{1,h_{k+1}} - u_{1,h_k}), & \forall \nu \in W. \end{cases} \quad (3.19)$$

Then combining Lemma 3.1, Corollary 3.1, (3.19) and

$$\|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a,$$

we have the following inequality

$$\begin{aligned} \|w_{h_{k+1}}\|_a + |\gamma| &\lesssim \|\bar{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k}\| \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b + \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_b^2 \\ &\lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \end{aligned} \quad (3.20)$$

It means the following inequality holds

$$\|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \quad (3.21)$$

Combining the above inequality (3.21), the definition (3.14),  $\|\bar{u}_{1,h_{k+1}}\|_b = 1$  and  $\|\hat{u}_{1,h_{k+1}}\|_b \geq \|\bar{u}_{1,h_{k+1}}\|_b - \|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_k}\|_b$  having a lower bound from zero, we have the following inequalities

$$\begin{aligned} &\|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a \\ &\leq \left\| \bar{u}_{1,h_{k+1}} - \frac{\bar{u}_{1,h_{k+1}}}{\|\hat{u}_{1,h_{k+1}}\|_b} \right\|_a + \frac{\|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_a}{\|\hat{u}_{1,h_{k+1}}\|_b} \\ &\leq \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\hat{u}_{1,h_{k+1}}\|_b} \left| \|\bar{u}_{1,h_{k+1}}\|_b - \|\hat{u}_{1,h_{k+1}}\|_b \right| + \frac{\|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_a}{\|\hat{u}_{1,h_{k+1}}\|_b} \\ &\leq \frac{\|\bar{u}_{1,h_{k+1}}\|_a}{\|\hat{u}_{1,h_{k+1}}\|_b} \|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_b + \frac{\|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_a}{\|\hat{u}_{1,h_{k+1}}\|_b} \\ &\lesssim \|\bar{u}_{1,h_{k+1}} - \hat{u}_{1,h_{k+1}}\|_a \lesssim \|\bar{u}_{1,h_{k+1}} - u_{1,h_k}\|_a^2. \end{aligned}$$

This is the desired result (3.16). Furthermore, from (3.16) and Lemma 3.1, the other desired result (3.17) can be obtained easily and the proof is complete.  $\square$

**Remark 3.1.** Theorem 3.2 shows that the Newton iteration method has second order convergence rate when the initial approximation has enough accuracy. We also would like to say that Theorem 3.2 and its proof also give the analysis for the algebraic eigenvalue problems by the Newton iteration method.

## 4 Multilevel iteration method

In this section, we introduce a type of multilevel scheme based on the *One Newton Iteration Step* defined by Algorithm 3.1. The proposed multigrid method can obtain eigenpair approximation with the optimal accuracy and with much smaller computational work compared with solving the eigenvalue problem directly in the finest finite element space.

Before introducing the multigrid scheme, we define a sequence of triangulations  $\mathcal{T}_{h_k}$  of  $\Omega$ . Suppose  $\mathcal{T}_{h_1}$  is given and let  $\mathcal{T}_{h_k}$  be obtained from  $\mathcal{T}_{h_{k-1}}$  via regular refinement (produce  $\beta^d$  subelements) such that

$$h_k = \frac{1}{\beta} h_{k-1}.$$

Based on this sequence of meshes, we construct the corresponding nested linear finite element spaces such that

$$V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n}, \quad (4.1)$$

and the following relation of approximation errors hold

$$\frac{1}{\beta} \eta_a(h_{k-1}) \leq C_7 \eta_a(h_k), \quad \frac{1}{\beta} \delta_{h_{k-1}}(\lambda) \leq C_7 \delta_{h_k}(\lambda), \quad k = 2, \dots, n. \quad (4.2)$$

From the error estimate results in Proposition 2.1, we have

$$\|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a \leq C_8 \delta_{h_k}(\lambda_1), \quad k = 1, \dots, n-1, \quad (4.3)$$

where the constant  $C_8$  is a constant independent of the mesh size  $h_k$ .

**Algorithm 4.1.** *Multilevel Eigenvalue Iteration Scheme*

1. Construct a series of nested finite element spaces  $V_{h_1}, V_{h_2}, \dots, V_{h_n}$  such that (4.1) and (4.2) hold.

2. Solve the following eigenvalue problem:

Find  $(\lambda_{1,h_1}, u_{1,h_1}) \in \mathbb{R} \times V_{h_1}$  such that  $b(u_{1,h_1}, u_{1,h_1}) = 1$  and

$$a(u_{1,h_1}, v_{h_1}) = \lambda_{1,h_1} b(u_{1,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}. \quad (4.4)$$

3. Do  $k = 1, \dots, n-1$

Obtain a new eigenpair approximation  $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$  by a Newton iteration step

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \text{Newton\_Iteration}(\lambda_{1,k}, u_{1,h_k}, V_{h_{k+1}}). \quad (4.5)$$

End do

Finally, we obtain an eigenpair approximation  $(\lambda_{1,h_n}, u_{1,h_n}) \in \mathbb{R} \times V_{h_n}$ .

**Theorem 4.1.** Assume  $h_1$  is small enough such that  $(\lambda_{1,h_1}, u_{1,h_1})$  satisfies conditions (3.6) and (3.7). After implementing Algorithm 4.1, the resultant eigenpair approximation  $(\lambda_{1,h_n}, u_{1,h_n})$  has the following error estimates

$$\|u_{1,h_n} - \bar{u}_{1,h_n}\|_a \leq \delta_{h_n}(\lambda_1), \quad (4.6)$$

$$|\lambda_{1,h_n} - \bar{\lambda}_{1,h_n}| \leq C_9 \delta_{h_n}^2(\lambda_1), \quad (4.7)$$

when the mesh size  $h_1$  is small enough.

Besides, there exists an eigenfunction  $u_1$  of (2.1) corresponding to  $\lambda_1$  such that the following final convergence results hold

$$\|u_1 - u_{1,h_n}\|_a \leq 2\delta_{h_n}(\lambda_1), \quad (4.8)$$

$$|\lambda_1 - \lambda_{1,h_n}| \leq 2C_{10} \delta_{h_n}^2(\lambda_1). \quad (4.9)$$

*Proof.* Let us prove (4.6) by the method of induction. First, it is obvious that (4.6) holds for  $n = 1$  according to (4.4). Then we assume that (4.6) holds for  $n = k$ . It means we have the following estimate

$$\|\bar{u}_{1,h_k} - u_{1,h_k}\|_a \leq \delta_{h_k}(\lambda_1). \quad (4.10)$$

Now let us consider the case of  $n = k + 1$ . Combining (4.3), (4.10) and the triangle inequality leads to the following estimates

$$\begin{aligned} \|\bar{u}_{1,h_{k+1}} - u_{1,h_{k+1}}\|_a &\leq C_5 \|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2 \\ &\leq 2C_5 \|u_{1,h_k} - \bar{u}_{1,h_k}\|_a^2 + 2C_5 \|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^2 \\ &\leq 2C_5 \delta_{h_k}^2(\lambda_1) + 2C_5 C_8^2 \delta_{h_k}^2(\lambda_1) \\ &\leq 2C_5 (1 + C_8^2) \delta_{h_k}^2(\lambda_1) \\ &= \left(2\beta C_5 (1 + C_8^2) \delta_{h_k}(\lambda_1)\right) \frac{\delta_{h_k}(\lambda_1)}{\beta} \\ &\leq \left(2\beta C_5 C_7 (1 + C_8^2) \delta_{h_k}(\lambda_1)\right) \delta_{h_{k+1}}(\lambda_1). \end{aligned} \quad (4.11)$$

This means that the result (4.6) also holds for  $n = k + 1$  if  $2\beta C_5 C_7 (1 + C_8^2) \delta_{h_k}(\lambda_1) < 1$ . Thus we prove the desired result (4.6). From Lemma 3.1 and (4.6), we can obtain the desired result (4.7). Finally, (4.8) and (4.9) can be proved from (2.7), (2.9), (4.6), (4.7) and the triangle inequality.  $\square$

## 4.1 Multi eigenvalues

Now, we turn to extend the Newton iteration (3.2) for solving one eigenvalue to the corresponding version for multi eigenvalues (include simple and multiple eigenvalues). Assume that  $\lambda_m < \lambda_{m+1}$  and we have obtained the first  $m$  eigenpairs approximation  $\{(\mu_j, u_{0,j})\}_{j=1}^m$  to the problem (3.1), which satisfy

$$b(u_{0,i}, u_{0,j}) = \delta_{ij}, \quad i, j = 1, \dots, m,$$

where  $\mu_j$  is the Rayleigh quotient of  $u_{0,j}$ .

The Newton iteration method for (3.1) is to find  $(x_j, \tilde{u}_j) \in \mathbb{R}^m \times V$  ( $j = 1, \dots, m$ ) such that

$$\begin{cases} a(\tilde{u}_j, v) - \mu_j \cdot b(\tilde{u}_j, v) - \sum_{i=1}^m x_{ij} b(u_{0,i}, v) &= -\mu_j b(u_{0,j}, v), \quad \forall v \in V, \\ b(\tilde{u}_j, u_{0,i}) &= b(u_{0,j}, u_{0,i}), \quad \forall i = 1, \dots, m, \end{cases} \quad (4.12)$$

where  $x_{ij}$  is the  $i$ -th component of  $x_j$ .

Now, we come to prove (4.12) has only one solution for any  $j = 1, \dots, m$ . For this aim, we define the following bilinear forms

$$A_{\mu_j}(u, v) = a(u, v) - \mu_j b(u, v), \quad B(v, y) = -\sum_{i=1}^m y_i b(u_{0,i}, v), \quad (4.13)$$

where  $u \in V$ ,  $v \in V$ ,  $y \in W = \mathbb{R}^m$ .

Assume that  $f_{\mu_j} \in V'$ ,  $g_j \in W'$  are defined as

$$f_{\mu_j}(v) = -\mu_j b(u_{0,j}, v), \quad g_j(y) = -\sum_{i=1}^m y_i b(u_{0,i}, u_{0,j}).$$

We consider the following multi mixed problems: Find  $(x_j, \tilde{u}_j) \in \mathbb{R}^m \times V$ , ( $j = 1, \dots, m$ ), such that

$$\begin{cases} A_{\mu_j}(\tilde{u}_j, v) + B(v, x) &= f_{\mu_j}(v), \quad \forall v \in V, \\ B(\tilde{u}_j, y) &= g_j(y), \quad \forall y \in W. \end{cases} \quad (4.14)$$

Define  $\mathcal{K} = M(\lambda_1) \cup \dots \cup M(\lambda_m)$ . About the existence and uniqueness of problem (4.14), the following theorem holds.

**Theorem 4.2.** *Assume that there exists a decomposition of eigenspace  $\mathcal{K}$  satisfying  $\mathcal{K} = M(\lambda_1) \oplus \dots \oplus M(\lambda_m)$  such that  $u_{0,j}$  is an eigenfunction approximation to  $M(\lambda_j)$  ( $j = 1, \dots, m$ ). Then the bilinear forms defined in (4.13) satisfy the following conditions*

1. *There exists  $\alpha > 0$  such that*

$$A_{\mu_j}(v, v) \geq \alpha \|v\|_a^2, \quad \forall v \in V_0, \quad (4.15)$$

where  $V_0 = \{v | B(v, y) = 0, \forall y \in W\} = \{v | b(u_{0,i}, v) = 0, \forall i = 1, \dots, m\}$ .

2. *There exists  $\sigma > 0$  such that*

$$\sup_{v \in V} \frac{B(v, y)}{\|v\|_a} \geq \sigma \|y\|, \quad \forall y \in W, \quad (4.16)$$

where  $\|y\| := \max_{i \in \{1, \dots, m\}} |y_i|$ .

Based on these two conditions, for any  $j$  ( $j = 1, \dots, m$ ), the multi mixed equations (4.14) have only one solution.

*Proof.* We decompose  $u_{0,j}$  as  $u_{0,j} = w_{0,j} + w_{0,j}^\perp$  such that  $w_{0,j} \in M(\lambda_j)$  and  $w_{0,j}^\perp \perp_b w_{0,j}$ . Then  $\text{span}\{w_{0,1}, \dots, w_{0,m}\}$  is an orthonormal basis of eigenspace  $\mathcal{K}$ .

Since  $u_{0,j}$  ( $\|u_{0,j}\|_b = 1$ ) is an eigenfunction approximation to  $M(\lambda_j)$  with sufficiently small error, there is a small enough number  $\delta$  such that

$$\|u_{0,j} - w_{0,j}\|_a \leq \delta, \quad u_{0,j} - w_{0,j} \perp_b \text{span}\{w_{0,j}\}, \quad j = 1, \dots, m. \quad (4.17)$$

From Lemma 3.1, we also have

$$|\mu_j - \lambda_j| \leq C\delta^2, \quad j = 1, \dots, m. \quad (4.18)$$

Since (4.17) and  $\|u_{0,j}\|_b^2 = \|w_{0,j}\|_b^2 + \|w_{0,j}^\perp\|_b^2$ ,  $w_{0,j}^\perp$  and  $w_{0,j}$  have estimates

$$\|w_{0,j}^\perp\|_b \leq C\|w_{0,j}^\perp\|_a \leq C\delta, \quad \|w_{0,j}\|_b \geq 1 - C\delta, \quad j = 1, \dots, m.$$

Similarly, we also do decomposition  $v \in V_0$  as

$$v = v_1 + \dots + v_m + v^* = v_j + v_j^\perp, \quad j = 1, \dots, m$$

satisfying

$$v^* \perp_b \mathcal{K}, \quad v_j \in \text{span}\{w_{0,j}\}, \quad v_j^\perp \perp_b \text{span}\{w_{0,j}\}.$$

According to the definition of  $v \in V_0$ , i.e.,  $b(w_{0,j} + w_{0,j}^\perp, v_j + v_j^\perp) = 0$ , we have

$$\begin{aligned} \|v_j\|_b \|w_{0,j}\|_b &= |b(v_j, w_{0,j})| = |-b(v_j^\perp, w_{0,j}^\perp)| = |b(v, w_{0,j}^\perp)| \\ &\leq C\delta \|v\|_b, \quad j = 1, \dots, m. \end{aligned}$$

Furthermore,

$$\|v_j\|_b \leq \frac{C\delta}{1 - C\delta} \|v\|_b \leq C\delta \|v\|_b, \quad j = 1, \dots, m. \quad (4.19)$$

From (4.19) and the property  $\|v\|_b^2 = \|v_1\|_b^2 + \dots + \|v_m\|_b^2 + \|v^*\|_b^2$ , the following estimates hold

$$\begin{aligned} b(v, v) &= b(v_1, v_1) + \dots + b(v_m, v_m) + b(v^*, v^*) \\ &\leq mC\delta^2 b(v, v) + \frac{1}{\lambda_{m+1}} a(v^*, v^*) \\ &\leq mC\delta^2 b(v, v) + \frac{1}{\lambda_{m+1}} a(v, v). \end{aligned}$$

Thus we have the following inequality

$$b(v, v) \leq \frac{1}{\lambda_{m+1}(1 - mC\delta^2)} a(v, v). \quad (4.20)$$

From (4.18), (4.20) and the definition of  $A_{\mu_j}(\cdot, \cdot)$ , the following inequalities hold

$$\begin{aligned} a(v, v) - \mu_j b(v, v) &\geq \left(1 - \frac{\mu_j}{\lambda_{m+1}(1 - mC\delta^2)}\right) a(v, v) \\ &\geq \frac{\lambda_{m+1}(1 - mC\delta^2) - \mu_j}{\lambda_{m+1}(1 - mC\delta^2)} a(v, v) \\ &\geq \frac{\lambda_{m+1} - \lambda_j - C\delta^2}{\lambda_{m+1}(1 - mC\delta^2)} a(v, v). \end{aligned}$$

It means (4.15) holds for  $\alpha = (\lambda_{m+1} - \lambda_j - C\delta^2)/(\lambda_{m+1}(1 - mC\delta^2)) > 0$  ( $j = 1, \dots, m$ ), when  $\delta$  is small enough.

Now, we come to prove (4.16). Assume that the index  $s$  satisfies  $\|y\| = |y_s|$ . From  $b(u_{0,i}, u_{0,j}) = \delta_{ij}$  ( $i, j = 1, \dots, m$ ) and the definition of  $B(\cdot, \cdot)$  and  $\mu_j$ , taking  $v = -\text{sign}(y_s)u_{0,s}$ , we have

$$\sup_{v \in V} \frac{B(v, y)}{\|v\|_a} \geq \frac{|y_s|b(u_{0,s}, u_{0,s})}{\|u_{0,s}\|_a} = \frac{\|y\|}{\mu_s} \geq \frac{\|y\|}{\mu} > 0, \quad \forall y \in W,$$

where  $\mu = \max_{t \in \{1, 2, \dots, m\}} \{\mu_t\}$ . It means that (4.16) holds for

$$\sigma = \frac{1}{\mu}.$$

From the theory for the mixed finite element method [4], there exists only one solution for the equations (4.14) for any  $j = 1, \dots, m$ .  $\square$

## 4.2 Multilevel iteration for multi eigenvalues

Based on the discussion in the last subsection, we extend the one iteration step to improve given eigenpairs approximation to the first  $m$  given eigenpair approximations. Assume we have obtained  $m$  eigenpairs approximation  $(\lambda_{i,h_k}, u_{i,h_k}) \in \mathbb{R} \times V_{h_k}$  with  $\|u_{i,h_k}\|_b = 1$  ( $i = 1, \dots, m$ ). Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximation  $(\lambda_{i,h_k}, u_{i,h_k})$ . Let  $V_{h_{k+1}} \subset V$  be a finer finite element space such that  $V_{h_k} \subset V_{h_{k+1}}$ . Based on this finer finite element space, we define the following one Newton iteration step for multi eigenvalues. We can state the following version of *Multilevel Eigenvalue Iteration Scheme* for  $m$  eigenvalues.

Similarly, we first give a type of *One Iteration Step for Multi Eigenvalues* for the given eigenpair approximations  $\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^m$ .

**Algorithm 4.2.** *One Newton Iteration Step for Multi Eigenvalues*

1. Do  $i = 1, \dots, m$   
 Find  $(x_{i,h_{k+1}}, \tilde{u}_{i,h_{k+1}}) \in \mathbb{R} \times V_{h_{k+1}}$  such that
 
$$\begin{cases} a(\tilde{u}_{i,h_{k+1}}, v_{h_{k+1}}) - \lambda_{i,h_k} b(\tilde{u}_{i,h_{k+1}}, v_{h_{k+1}}) - \sum_{s=1}^m x_{si,h_{k+1}} b(u_{s,h_k}, v_{h_{k+1}}) \\ \quad = -\lambda_{i,h_k} b(u_{i,h_k}, v_{h_{k+1}}), & \forall v_{h_{k+1}} \in V_{h_{k+1}}, \\ b(\tilde{u}_{i,h_{k+1}}, u_{j,h_k}) = \delta_{ij}, & \forall j = 1, \dots, m, \end{cases} \quad (4.21)$$
 where  $x_{si,h_{k+1}}$  is the  $s$ -th component of  $x_{i,h_{k+1}}$ .  
 End Do
2. Build a finite dimensional space  $\tilde{V}_{h_{k+1}} = \text{span}\{\tilde{u}_{1,h_{k+1}}, \dots, \tilde{u}_{m,h_{k+1}}\}$  and solve the following eigenvalue problem:  
 Find  $(\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}) \in \mathbb{R} \times \tilde{V}_{h_{k+1}}$ ,  $i = 1, 2, \dots, m$ , such that  $b(u_{i,h_{k+1}}, u_{i,h_{k+1}}) = 1$  and

$$a(u_{i,h_{k+1}}, v_{h_{k+1}}) = \lambda_{i,h_{k+1}} b(u_{i,h_{k+1}}, v_{h_{k+1}}), \quad \forall v_{h_{k+1}} \in \tilde{V}_{h_{k+1}}.$$

We summarize above two steps into

$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^m = \text{Newton\_Iteration}(\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^m, V_{h_{k+1}}).$$

Based on Algorithm 4.2, we come to give the corresponding multilevel correction method.

**Algorithm 4.3.** *Multilevel Eigenvalue Iteration Scheme for Multi Eigenvalues*

1. Construct a series of nested finite element spaces  $V_{h_1}, V_{h_2}, \dots, V_{h_n}$  such that (4.1) and (4.2) hold.
2. Solve the eigenvalue problem in the initial finite element space  $V_{h_1}$ :  
 Find  $(\lambda_{h_1}, u_{h_1}) \in \mathbb{R} \times V_{h_1}$  such that  $b(u_{h_1}, u_{h_1}) = 1$  and

$$a(u_{i,h_1}, v_{h_1}) = \lambda_{i,h_1} b(u_{i,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.$$

Choose the first  $m$  eigenpairs  $\{\lambda_{i,h_1}, u_{i,h_1}\}_{i=1}^m$  which approximate the desired eigenpairs.

3. Do  $k = 1, \dots, n - 1$

Obtain new eigenpair approximations  $\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^m \in \mathbb{R} \times V_{h_{k+1}}$  by the one Newton iteration step defined in Algorithm 4.2

$$\{\lambda_{i,h_{k+1}}, u_{i,h_{k+1}}\}_{i=1}^m = \text{Newton\_Iteration}(\{\lambda_{i,h_k}, u_{i,h_k}\}_{i=1}^m, V_{h_{k+1}}).$$

End do

Finally, we obtain  $m$  eigenpair approximations  $\{\lambda_{i,h_n}, u_{i,h_n}\}_{i=1}^m \in \mathbb{R} \times V_{h_n}$ .

In Algorithm 4.2, the parallel computation can be used to solve (4.21) for different  $i$ . The analysis of the scheme for multi eigenvalues will be given in our future work.



## 5 Work estimate of multilevel eigenvalue iteration scheme

In this section, we turn our attention to the estimate of computational work for Algorithm 4.1 (Algorithm 4.3). We will show that Algorithm 4.1 (Algorithm 4.3) makes solving the eigenvalue problem need almost the optimal computational work if solving the linear equation (3.13) needs only the linear computational work.

First, we investigate the dimension of each level linear finite element space as  $N_k := \dim V_{h_k}$ . Then the following property holds

$$N_k \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, 2, \dots, n. \quad (5.1)$$

**Theorem 5.1.** *Assume solving the eigenvalue problem in the coarse space  $V_{h_1}$  needs work  $\mathcal{O}(M_{h_1})$  and the work for solving the linear equation (3.13) (when  $m > 1$ , for each  $i$ , using parallel technique to solve (4.21)) in each level space  $V_{h_k}$  is only  $\mathcal{O}(N_k)$  for  $k = 2, \dots, n$ . Then the work involved in Algorithm 4.1 (Algorithm 4.3) is  $\mathcal{O}(N_n + M_{h_1})$ . Furthermore, the complexity will be  $\mathcal{O}(N_n)$  provided  $M_{h_1} \leq N_n$ .*

*Proof.* Let  $W_k$  denote the work of the iteration step defined in Algorithm 3.1 (Algorithm 4.2 in each computing node) in the  $k$ -th finite element space  $V_{h_k}$  for  $k = 2, \dots, n$ . From the iteration definition in Algorithm 3.1 (Algorithm 4.2), we have

$$W_k = \mathcal{O}(N_k), \quad \text{for } k = 2, \dots, n. \quad (5.2)$$

Iterating (5.2) and using the fact (5.1), the following estimates hold

$$\begin{aligned} \text{Total work} &= \sum_{k=1}^n W_k = \mathcal{O}\left(M_{h_1} + \sum_{k=2}^n N_k\right) = \mathcal{O}\left(M_{h_1} + \sum_{k=2}^n N_k\right) \\ &= \mathcal{O}\left(M_{h_1} + \sum_{k=2}^n \left(\frac{1}{\beta}\right)^{d(n-k)} N_n\right) = \mathcal{O}(N_n + M_{h_1}). \end{aligned} \quad (5.3)$$

This is the desired estimate  $\mathcal{O}(N_n + M_{h_1})$  for the computational work and the one  $\mathcal{O}(N_n)$  can be derived with the condition  $M_{h_1} \leq N_n$ .  $\square$

## 6 Numerical results

In this section, two numerical examples are presented to illustrate the efficiency of the multilevel iteration scheme proposed in this paper.

## 6.1 Model eigenvalue problem

Here we give the numerical results of the multilevel iteration scheme for the Laplace eigenvalue problem on the two dimensional domain  $\Omega = (0, 1) \times (0, 1)$ . The sequence of finite element spaces is constructed by using linear element on the series of meshes which are produced by the regular refinement with  $\beta = 2$  (producing  $\beta^2$  subelements). In this example, we use two meshes which are generated by Delaunay method as the initial mesh  $\mathcal{T}_{h_1}$  ( $H = h_1$ ) to produce two sequences of finite element spaces for investigating the convergence behaviors. Figure 1 shows the corresponding initial meshes: one is coarse and the other is fine.

Algorithm 4.1 is applied to solve the eigenvalue problem. For comparison, we also solve the eigenvalue problem by the direct method.

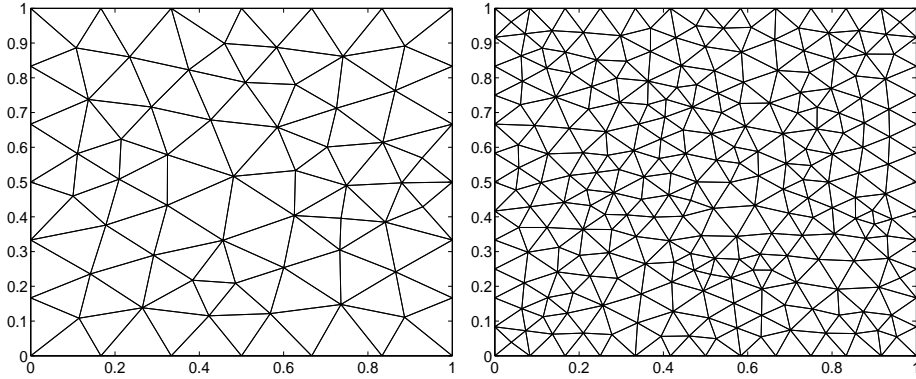


Figure 1: The initial coarse  $H = 1/6$  and fine  $H = 1/12$  meshes for Example 1

Figure 2 gives the corresponding numerical results for the first eigenvalue  $\lambda_1 = 2\pi^2$  and the corresponding eigenfunction on the two initial meshes illustrated in Figure 1. From Figure 2, we find the multilevel iteration scheme can obtain the optimal error estimates as same as the direct eigenvalue solving method for the eigenvalue and the corresponding eigenfunction approximations.

We also check the convergence behavior for multi eigenvalue approximations with Algorithm 4.1. Here the first six eigenvalues  $\lambda = 2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2$  are investigated. We adopt the meshes in Figure 1 as the initial ones and the corresponding numerical results are shown in Figure 3. Figure 3 also exhibits the optimal convergence rate of the multilevel iteration scheme.

## 6.2 More general eigenvalue problem

Here we give the numerical results of the multilevel iteration scheme for solving a more general eigenvalue problem on the unit square domain  $\Omega = (0, 1) \times (0, 1)$ .

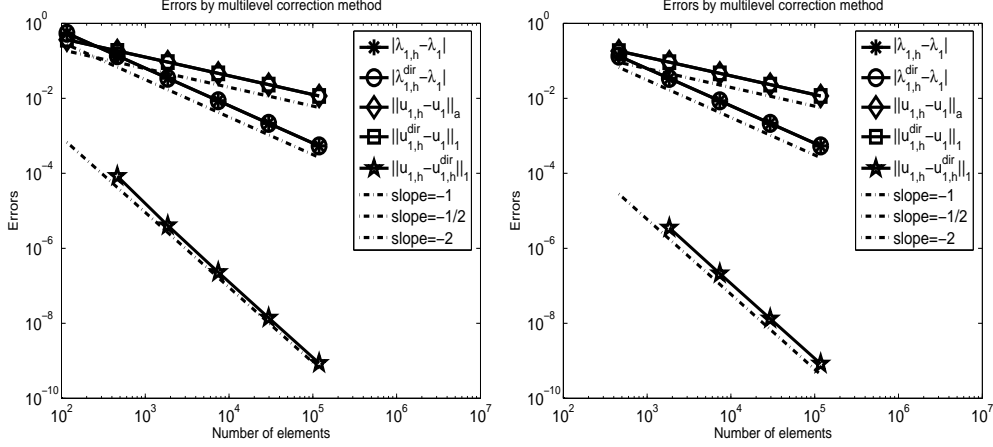


Figure 2: The errors of the multilevel iteration algorithm for the first eigenvalue  $2\pi^2$  and the corresponding eigenfunction, where  $u_h^{\text{dir}}$  and  $\lambda_h^{\text{dir}}$  denote the eigenfunction and eigenvalue approximation by direct eigenvalue solving (The left subfigure is for the coarse initial mesh in the left of Figure 1 and the right one for the fine initial mesh in the right of Figure 1)

Find  $(\lambda, u)$  such that

$$\begin{cases} -\nabla \cdot \mathcal{A} \nabla u + \phi u &= \lambda \rho u, & \text{in } \Omega, \\ u &= 0, & \text{on } \partial\Omega, \\ \int_{\Omega} \rho u^2 d\Omega &= 1, \end{cases} \quad (6.1)$$

where

$$\mathcal{A} = \begin{pmatrix} 1 + (x_1 - \frac{1}{2})^2 & (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) \\ (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) & 1 + (x_2 - \frac{1}{2})^2 \end{pmatrix},$$

$\phi = e^{(x_1 - \frac{1}{2})(x_2 - \frac{1}{2})}$  and  $\rho = 1 + (x_1 - \frac{1}{2})(x_2 - \frac{1}{2})$ .

We first solve the eigenvalue problem (6.1) in the linear finite element space on the coarse mesh  $\mathcal{T}_{h_1}$ . Then refine the mesh by the regular way to produce a series of meshes  $\mathcal{T}_{h_k}$  ( $k = 2, \dots, n$ ) with  $\beta = 2$  (connecting the midpoints of each edge) and solve the augmented mixed problem (3.13) in the finer linear finite element space  $V_{h_k}$  defined on  $\mathcal{T}_{h_k}$ .

In this example, we also use two coarse meshes which are shown in Figure 1 as the initial meshes to investigate the convergence behaviors. Since the exact solution is unknown, we choose an adequately accurate eigenvalue approximations with the extrapolation method (see, e.g., [9]) as the exact eigenvalue. Figure 4 gives the corresponding numerical results for the first six eigenvalue approximations and their corresponding eigenfunction approximations. Here we also compare the numerical results with the direct algorithm. Figure 4 also exhibits the optimal convergence rate of Algorithm 4.1.

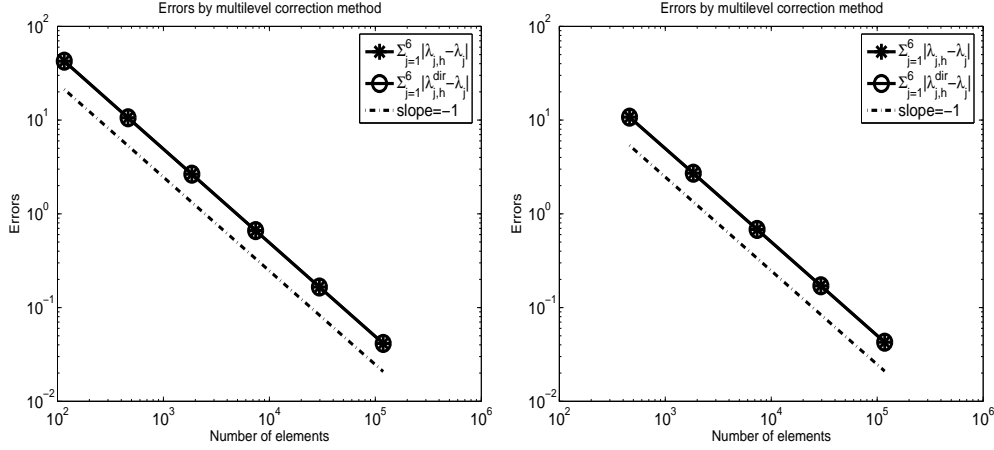


Figure 3: The errors of the multilevel iteration algorithm for the first six eigenvalues on the unit square (The left subfigure is for the coarse initial mesh in the left of Figure 1 and the right one for the fine initial mesh in the right of Figure 1)

## 7 Concluding remarks

In this paper, we propose a type of multilevel method for eigenvalue problems based on the Newton iteration scheme. In this type of iteration method, solving eigenvalue problem on the finest finite element space is decomposed into solving a small scale eigenvalue problem in a coarse initial space and solving a sequence of augmented linear problems, derived by Newton iteration step in the corresponding sequence of finite element spaces. The proposed scheme improves the overall efficiency of eigenvalue problem solving by the finite element method.

The quadratic convergence property of Newton's method improves the accuracy of the numerical solution. On the other hand, the multilevel technique overcomes the sensitivity of initial guess of Newton scheme.

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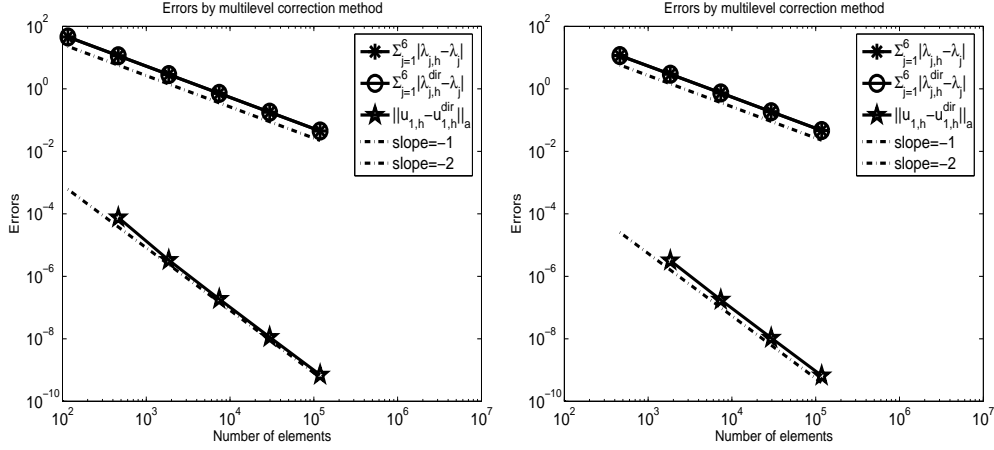


Figure 4: The errors of the multilevel iteration algorithm for the first six eigenvalues and the corresponding first eigenfunction, where  $u_h^{\text{dir}}$  and  $\lambda_h^{\text{dir}}$  denote the eigenfunction and eigenvalue approximation by direct eigenvalue solving (The left subfigure is for the coarse initial mesh in the left of Figure 1 and the right one for the fine initial mesh in the right of Figure 1)

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